Stress, Strain, and the Equation of Motion

ESS 314 Geophysics · University of Washington

Week 1, Lecture 3 · April 7, 2026

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By the end of this lecture…

- **[LO-3.1]** *Define* the stress and strain tensors; identify normal vs. shear components
- **[LO-3.2]** *Relate* the four elastic moduli (E, K, μ, ν) to deformation geometries
- **[LO-3.3]** *Write* isotropic Hooke's law using Lamé parameters λ and μ
- **[LO-3.4]** *Derive* the equation of motion; identify $V_P = \sqrt{(\lambda + 2\mu)/\rho}$
- **[LO-3.5]** *Evaluate* the assumptions of linear elastic theory

Why Does the Ground Shake?

A Cascadia M9 earthquake will reach Seattle in **~90 seconds** as elastic waves traveling through rock

The wave speed — and the shaking intensity — depend on the **elastic properties of every rock layer** the wave passes through

Today: building the physics that connects rock stiffness to wave speed

Elastic Deformation: The Key Assumption

Elastic = material returns to its original shape after stress is removed

Linear elastic (Hookean) = stress ∝ strain

> Seismic strains are ~10^{-7} — far inside the Hookean regime. Fault zones, magma chambers, and the deep Earth are exceptions.

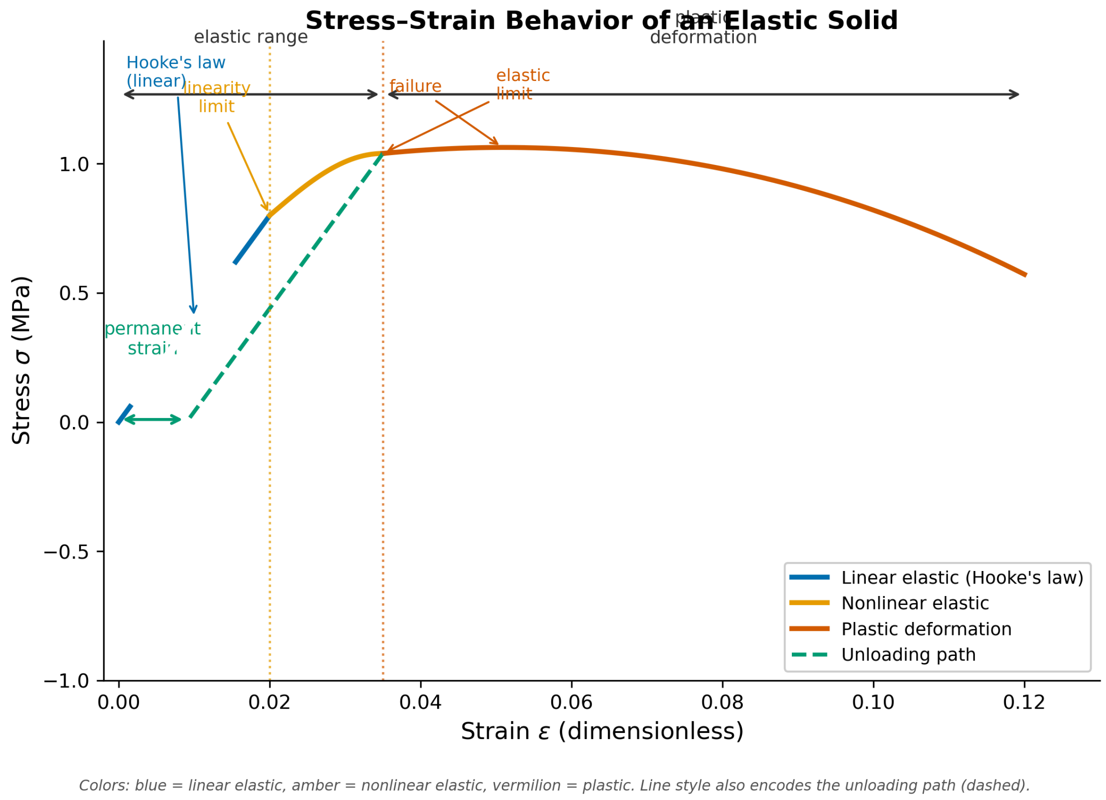


Figure 3.1. Seismic waves operate in the blue (linear elastic) region only. Python-generated — assets/scripts/fig_stress_strain_curve.py

Two Modes of Elastic Deformation

Volumetric (dilatational) strain θ

- Change in volume, no change in shape
- Resisted by **bulk modulus** K
- $\rightarrow$ P-waves

Shear strain ε_{ij} $(i \neq j)$

- Change in shape, no change in volume
- Resisted by **shear modulus** μ
- $\rightarrow$ S-waves

> Fluids: $\mu = 0$ $\rightarrow$ no resistance to shear $\rightarrow$ S-waves CANNOT travel in fluids

The Stress Tensor

$$\boldsymbol{\sigma} = \begin{pmatrix} \sigma_{xx} & \sigma_{xy} & \sigma_{xz} \\ \sigma_{yx} & \sigma_{yy} & \sigma_{yz} \\ \sigma_{zx} & \sigma_{zy} & \sigma_{zz} \end{pmatrix}$$

- **Diagonal** = normal stresses (compression / tension)
- **Off-diagonal** = shear stresses
- **Symmetric** ($\sigma_{ij} = \sigma_{ji}$) → 6 independent components
- **Force = Stress × Area:** $F_x = \sigma_{xx}\, A_x$ — stress is force per unit area on a surface

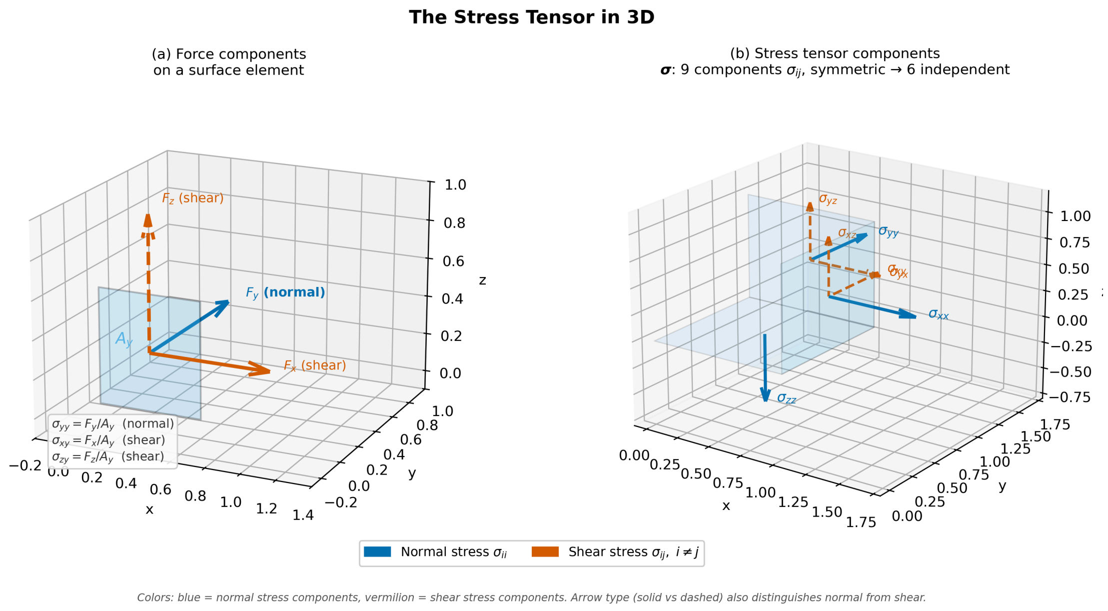


Figure 3.2. Normal stresses (blue) and shear stresses (vermilion) on a unit cube. Python-generated — assets/scripts/fig_stress_tensor.py

Three Modes of Strain

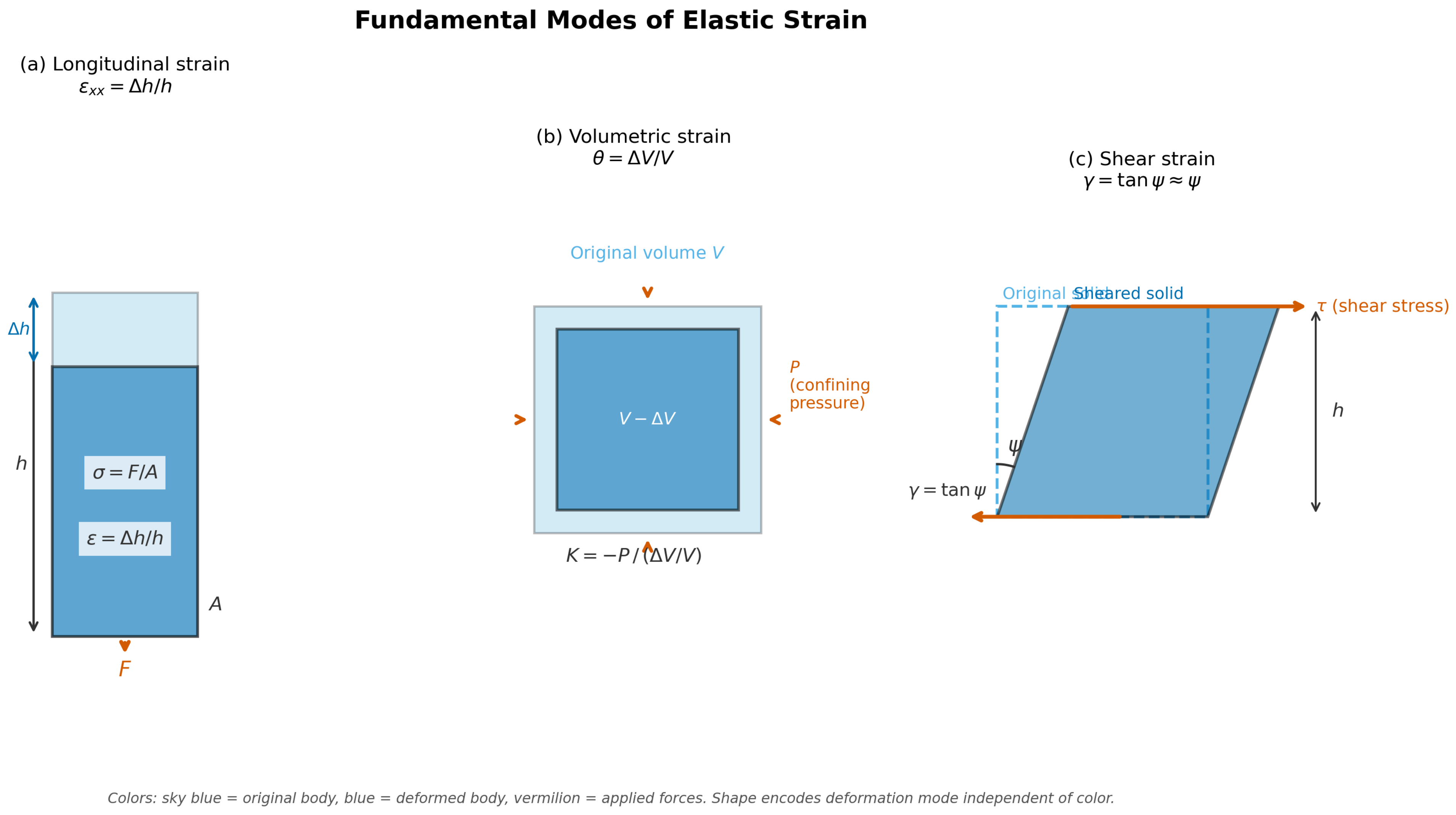


Figure 3.3. (a) Longitudinal ε_xx = Δh/h. (b) Volumetric θ = ΔV/V. (c) Shear γ = tan ψ. P-waves involve (a)+(b); S-waves involve (c). Python-generated — assets/scripts/fig_strain_types.py

The Strain Tensor

x = coordinate (fixed, meters) · $u(x)$ = displacement (how far that material point moved)

Strain = symmetric part of the displacement gradient:

$$\varepsilon_{ij} = \frac{1}{2}\left(\frac{\partial u_i}{\partial x_j} + \frac{\partial u_j}{\partial x_i}\right)$$

- Diagonal ($i = j$): **extension / compression** — modes (a) longitudinal and (b) volumetric above
- Off-diagonal ($i \neq j$): **angular distortion** — mode (c) shear above
- Factor of ½ excludes rigid-body rotation

Dilatation (volume change):

$$\theta = \varepsilon_{xx} + \varepsilon_{yy} + \varepsilon_{zz} = \nabla \cdot \mathbf{u}$$

Four Elastic Moduli

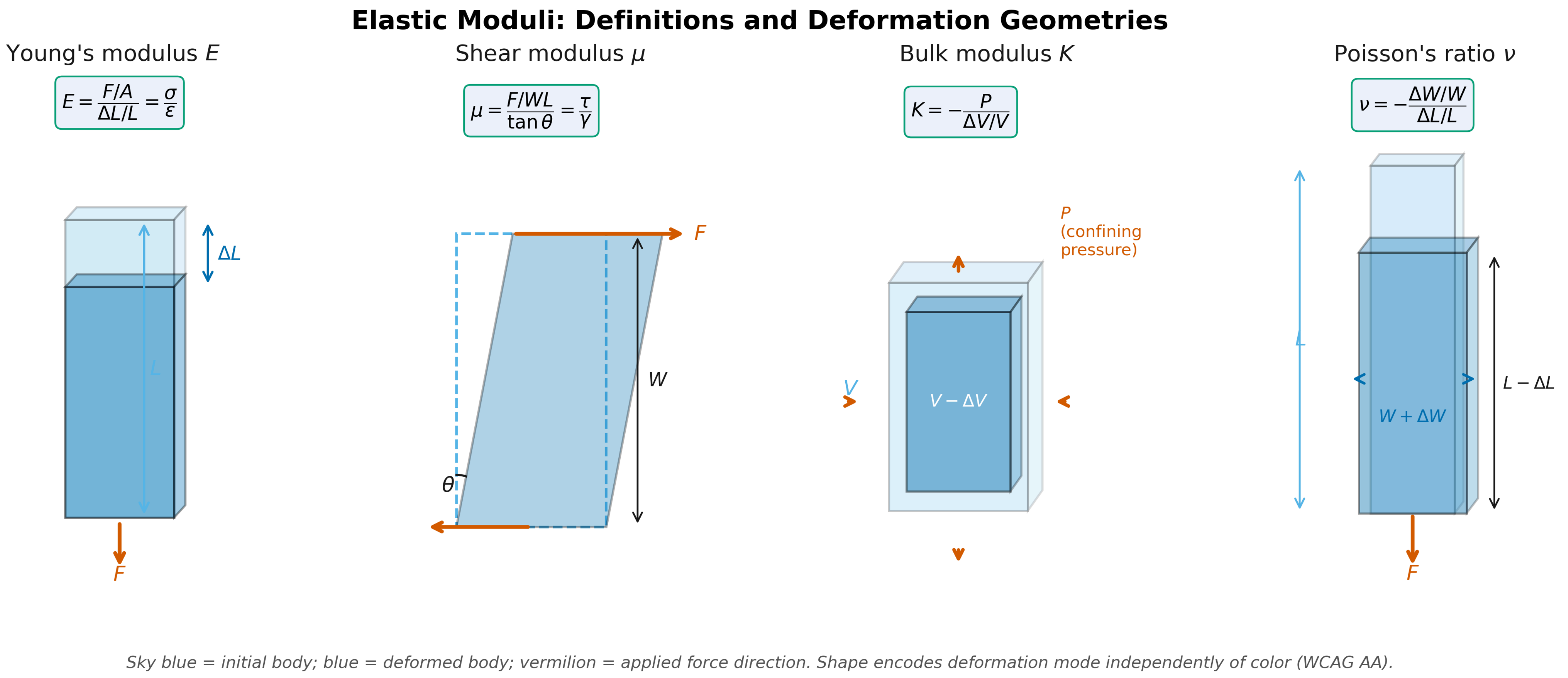


Figure 3.4. E (axial stiffness), μ (shear stiffness), K (bulk stiffness), ν (lateral/axial ratio). Python-generated — assets/scripts/fig_elastic_moduli.py

Elastic Moduli: Relationships

Any two moduli specify all others. Seismology uses **Lamé parameters** λ, μ:

$$\lambda = K - \tfrac{2}{3}\mu = \frac{\nu E}{(1+\nu)(1-2\nu)}$$

$$\mu = \frac{E}{2(1+\nu)} \qquad \text{(shear modulus = rigidity)}$$

Key conversions needed for seismology:

- V_P and V_S → λ, μ, ρ → E, K, ν
- Typical crustal granite: $\lambda \approx 30$ GPa, $\mu \approx 25$ GPa, $\rho \approx 2700$ kg/m^3

Isotropic Hooke's Law

For a homogeneous, isotropic, linear elastic solid:

$$\sigma_{ij} = \lambda\,\delta_{ij}\,\theta + 2\mu\,\varepsilon_{ij}$$

Term 1 ($\lambda\delta_{ij}\theta$): volume change drives normal stresses in ALL directions — the coupling term

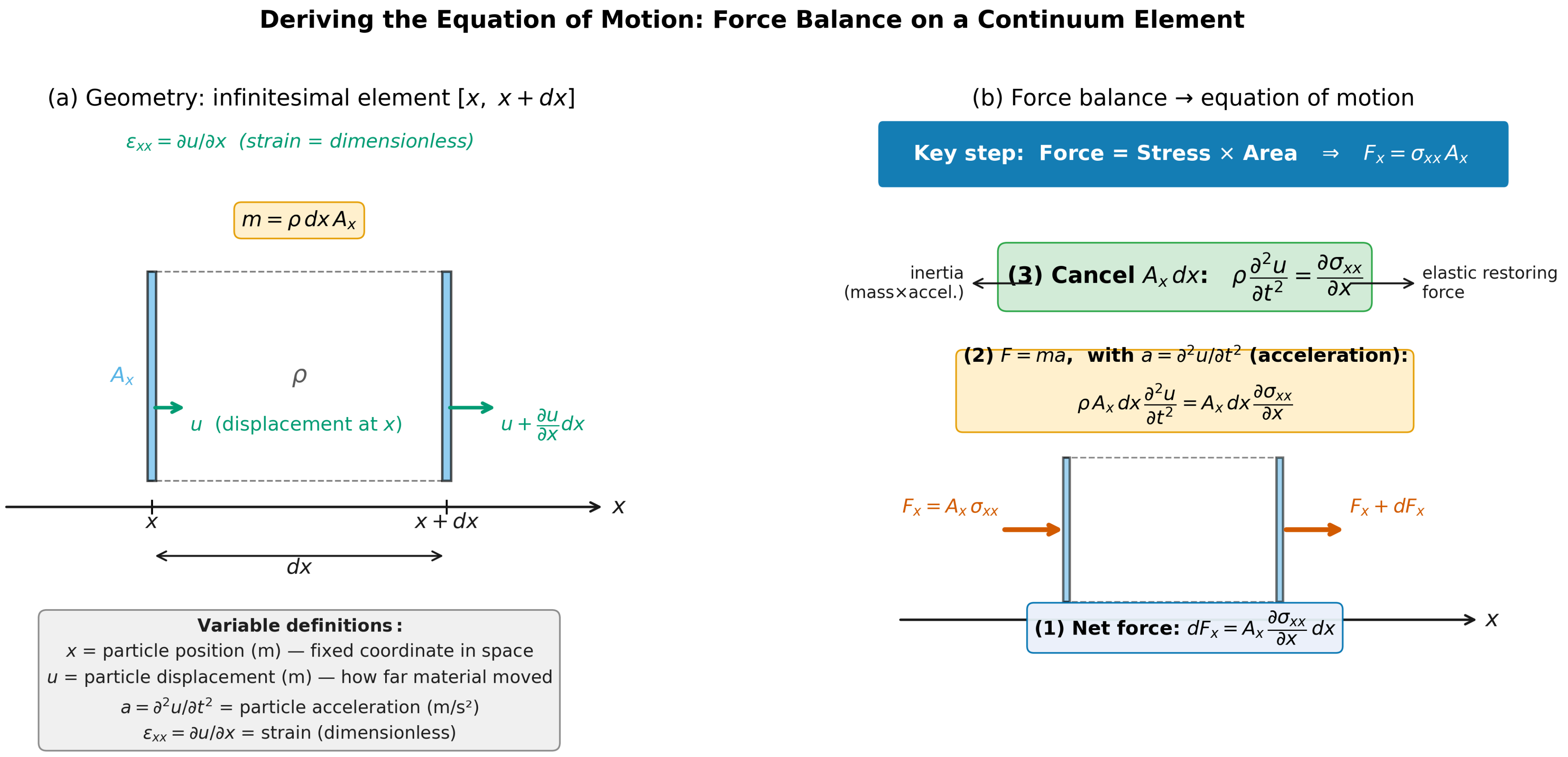
Term 2 ($2\mu\varepsilon_{ij}$): direct resistance to any strain (normal and shear)

Two parameters (λ, μ) because isotropy collapses 21 stiffness components to 2

Units: Pa · dimensionless + Pa · dimensionless = Pa = [stress] ✓

Force Balance on a Continuum Element

Apply **Force = Stress × Area** → Newton's $F = ma$ on an infinitesimal element of density ρ.



Colors: sky blue = element faces, vermilion = forces, green = displacements. Symbols and arrows both encode information (WCAG AA dual-coding).

The Equation of Motion → Wave Equation

Step 1 — Net force on element:

$$dF_x = A_x \frac{\partial \sigma_{xx}}{\partial x}\, dx$$

Step 2 — Newton's 2nd law ($F = ma$):

$$\rho \frac{\partial^2 u}{\partial t^2} = \frac{\partial \sigma_{xx}}{\partial x}$$

Step 3 — Substitute Hooke's law ($\sigma_{xx} = (\lambda + 2\mu)\partial u/\partial x$):

$$\rho \frac{\partial^2 u}{\partial t^2} = (\lambda + 2\mu) \frac{\partial^2 u}{\partial x^2}$$

$$\Rightarrow \quad V_P = \sqrt{\frac{\lambda + 2\mu}{\rho}}$$

Two Wave Speeds from One Equation

Wave	Equation	Speed
P (compressional)	$\rho\,\ddot{u} = (\lambda + 2\mu)\,u''$	$V_P = \sqrt{(\lambda + 2\mu)/\rho}$
S (shear)	$\rho\,\ddot{u} = \mu\,u''$	$V_S = \sqrt{\mu/\rho}$

Since $\lambda \geq 0$: $\lambda + 2\mu > \mu$ → $V_P > V_S$ **always**

Units: $\sqrt{\mathrm{Pa}/(\mathrm{kg}/\mathrm{m}^3)} = \sqrt{(\mathrm{kg}/\mathrm{m{\cdot}s}^2)/(\mathrm{kg}/\mathrm{m}^3)} = \mathrm{m}/\mathrm{s}$ ✓

> Stiffer rock → faster waves. Denser rock → slower waves.
> Their ratio sets the speed — not either quantity alone.

Worked Example: Granite

$\lambda = 30$ GPa, $\mu = 25$ GPa, $\rho = 2700$ kg/m³

$$V_P = \sqrt{\frac{(30 + 50) \times 10^9}{2700}} = \sqrt{2.96 \times 10^7} \approx 5443 \text{ m/s}$$

$$V_S = \sqrt{\frac{25 \times 10^9}{2700}} \approx 3043 \text{ m/s}$$

$$\frac{V_P}{V_S} = \sqrt{\frac{80}{25}} = \sqrt{3.2} \approx 1.79 \quad \Leftrightarrow \quad \nu \approx 0.27$$

Characteristic of **upper-crustal granite** ✓

Seismic Velocities: Typical Values

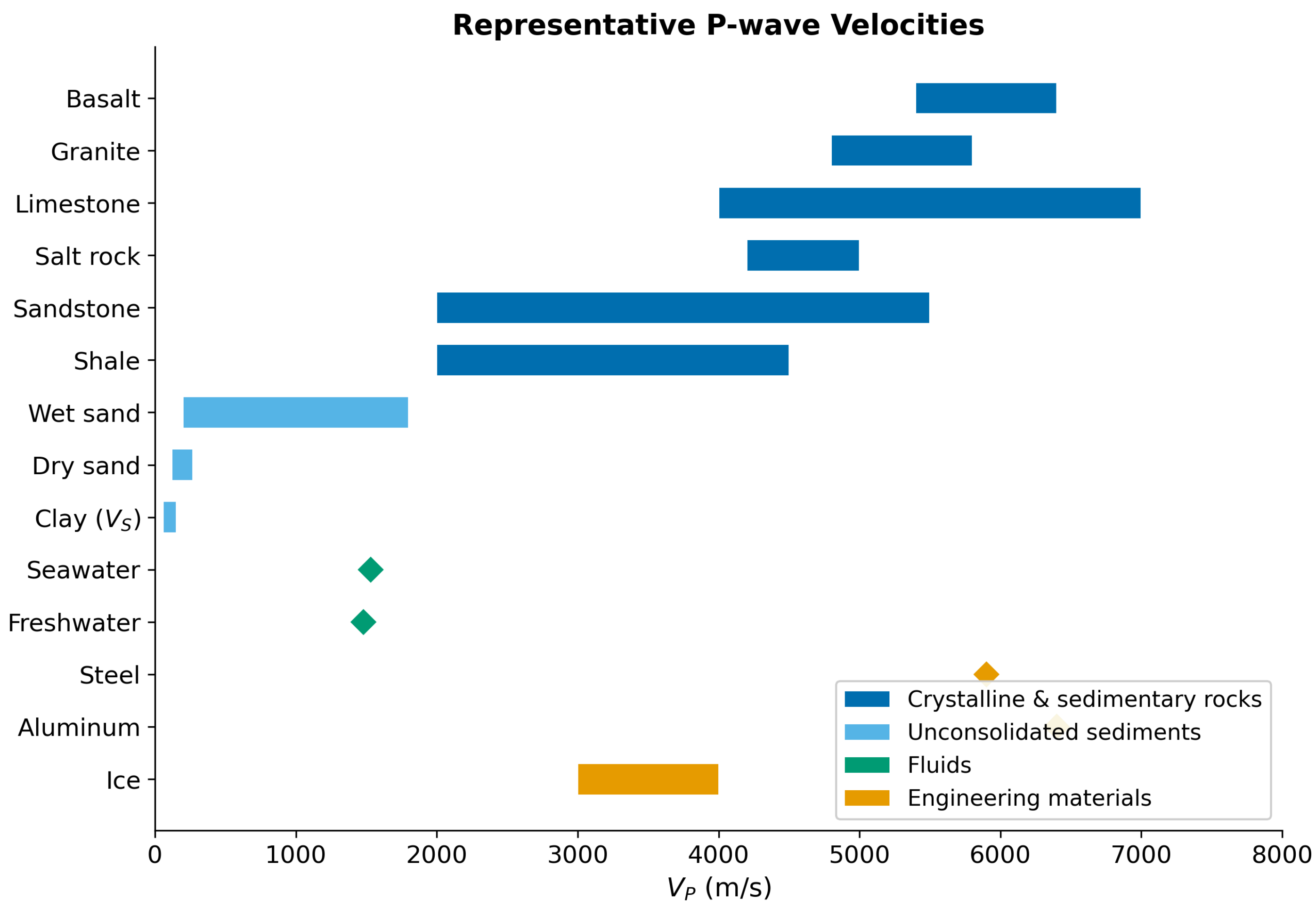


Figure 3.7. V_P spans nearly two orders of magnitude across Earth materials. Dry sand is ~100× slower than granite. Python-generated — assets/scripts/fig_seismic_velocities.py

The V_P / V_S Ratio as a Fluid Diagnostic

For $\nu = 0.25$ (typical crust): $V_P/V_S = \sqrt{3} \approx 1.73$

As $\nu \to 0.5$ (fluid saturation): $V_P/V_S \to \infty$

Seattle Basin example:

- $V_P \approx 1800$ m/s, $V_S \approx 300$ m/s
- $V_P/V_S = 6.0$, $\nu \approx 0.49$
- → water-saturated sediment

> High V_P/V_S = fluid. Low V_P/V_S = dry rock or gas sand.
> This is the single most useful seismic diagnostic in exploration and hazard.

AI Prompt Lab

Try this after class:

> *"Is V_P = sqrt(E/rho) or V_P = sqrt((λ+2μ)/rho) for seismic P-waves?"*

Both can be correct — but in different contexts. Evaluate whether the AI explains:

- $\sqrt{E/\rho}$: slender rod, uniaxial stress, free lateral expansion
- $\sqrt{(\lambda + 2\mu)/\rho}$: 3D bulk wave, constrained lateral deformation
- The conversion: $\lambda + 2\mu = E(1-\nu)/[(1+\nu)(1-2\nu)]$

If the AI gives only one answer without qualification → it has oversimplified.

Concept Check

1. A rock has $\lambda = 50$ GPa, $\mu = 30$ GPa, $\rho = 3100$ kg/m³. Calculate V_P, V_S, and ν. Show unit checks.
2. A sediment has $V_P = 1500$ m/s and $V_S = 50$ m/s. Calculate Poisson's ratio. What does this tell you about the physical state of the sediment?
3. The equation of motion was derived assuming the material is *homogeneous* and *isotropic*. Name one real-Earth situation where each assumption fails, and describe what new physics is needed.

Summary

Concept	Key Result
Elastic deformation	Hookean (linear elastic), small strains
Stress tensor	6 independent components ($\sigma_{ij} = \sigma_{ji}$)
Strain tensor	$\varepsilon_{ij} = \frac{1}{2}(\partial_j u_i + \partial_i u_j)$
Hooke's law	$\sigma_{ij} = \lambda \delta_{ij} \theta + 2\mu \varepsilon_{ij}$
Equation of motion	$\rho \ddot{u} = (\lambda + 2\mu) u''$ or $\mu u''$
P-wave speed	$V_P = \sqrt{(\lambda + 2\mu)/\rho}$
S-wave speed	$V_S = \sqrt{\mu/\rho}$

Next lecture: Wave types (P, S, Rayleigh, Love) and Snell's law