Wavefronts, Rays, and Snell's Law

ESS 314 Geophysics · University of Washington

Week 2, Lecture 5 · April 6, 2026

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By the end of this lecture…

- **[LO-5.1]** *Distinguish* wavefronts from rays and their geometric relationship
- **[LO-5.2]** *Apply* Huygens' principle to construct wavefronts at velocity contrasts
- **[LO-5.3]** *Derive* Snell's law from wavefront geometry
- **[LO-5.4]** *Define* the ray parameter p and explain why it is conserved
- **[LO-5.5]** *Derive* Snell's law from Fermat's principle of least time
- **[LO-5.6]** *Explain* why waves both reflect and refract at every interface
- **[LO-5.7]** *Describe* P–SV mode conversion and the generalized Snell's law
- **[LO-5.8]** *Define* acoustic impedance $Z = \rho V$ and compute normal-incidence R and T

The puzzle: bent ray paths

An earthquake occurs offshore of Westport, WA, at 30 km depth in the subducting Juan de Fuca plate.

The P-wave arrives at Olympia from an **unexpected direction** — steeper than the straight line from source to station.

The wave path was bent by velocity contrasts in the crust and upper mantle.

Today: the law that governs this bending — and two ways to derive it.

Wavefronts and rays

Wavefront = surface of constant phase (all points reached at the same time)

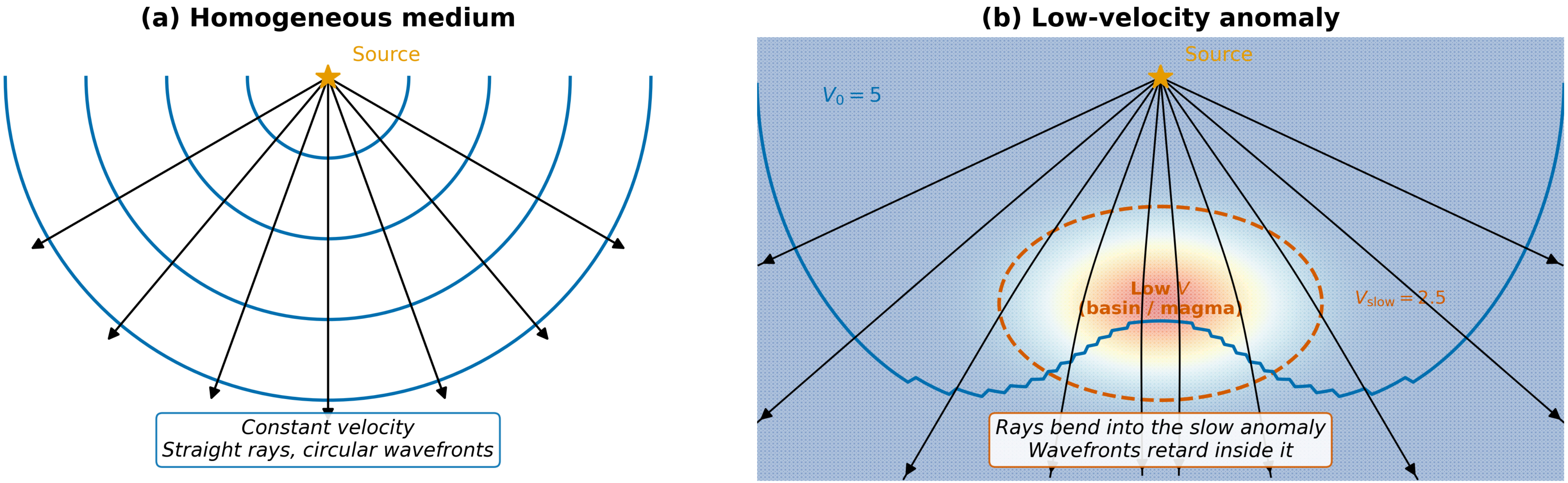
Ray = direction of energy propagation ($\perp$ to wavefront in isotropic media)

From a point source in a homogeneous medium:

- Wavefronts are **spherical** (3D) or circular (2D)
- Rays are **straight lines** radiating outward

At distances $\gg \lambda$: the wavefront looks locally flat $\rightarrow$ **plane wave** approximation

What happens in heterogeneous media?

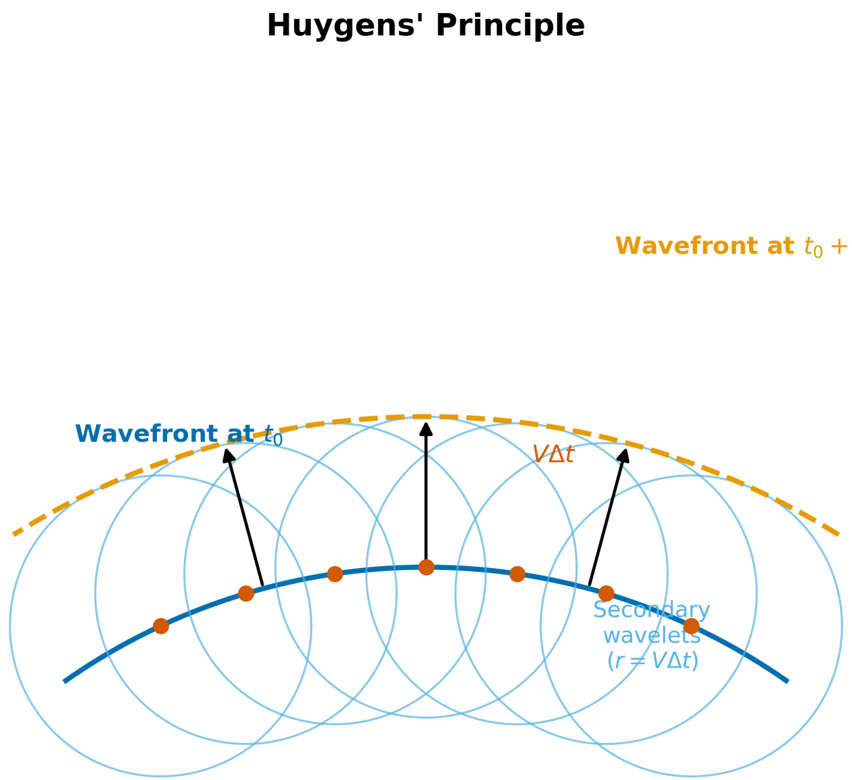


Key insight: The fast side of the wavefront advances farther → wavefront tilts → rays curve toward slower regions.

Huygens' principle (1678)

Every point on a wavefront acts as a **secondary point source**, emitting a spherical wavelet.

The new wavefront = **envelope** tangent to all secondary wavelets.



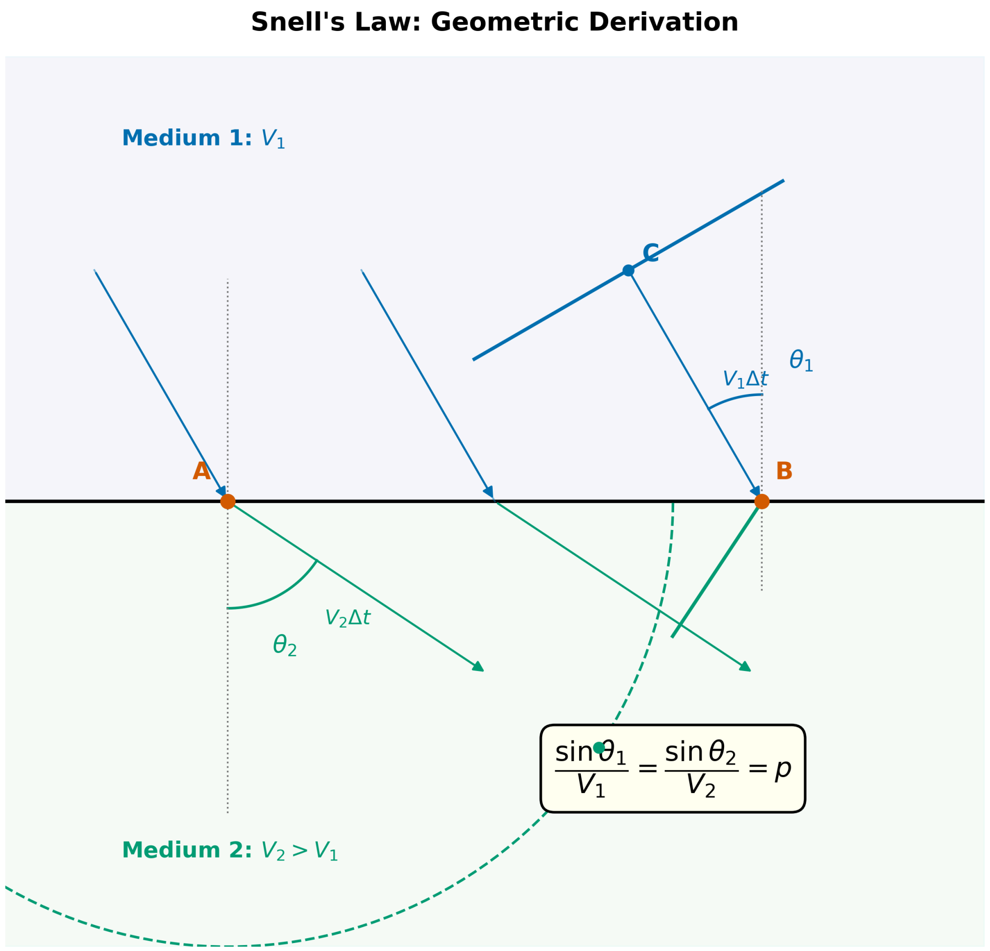
Huygens at a velocity contrast

At an interface between V_1 and $V_2 > V_1$:

- Wavelets in medium 2 are **larger** (radius $V_2 \Delta t > V_1 \Delta t$)
- The envelope tilts — wavefront changes direction
- The ray bends **away from the normal** into the faster medium

This is the physical mechanism behind Snell's law.

Deriving Snell's law: the geometry



Deriving Snell's law: the algebra

Two right triangles share hypotenuse AB:

$$\sin\theta_1 = \frac{BC}{AB} = \frac{V_1\,\Delta t}{AB}$$

$$\sin\theta_2 = \frac{AE}{AB} = \frac{V_2\,\Delta t}{AB}$$

Divide — Δt and AB cancel:

$$\boxed{\frac{\sin\theta_1}{V_1} = \frac{\sin\theta_2}{V_2} = p}$$

Snell's law: what it means

$$\frac{\sin\theta_1}{V_1} = \frac{\sin\theta_2}{V_2} = p$$

- $V_2 > V_1$: $\theta_2 > \theta_1$ → ray bends **away** from normal (into faster medium)
- $V_2 < V_1$: $\theta_2 < \theta_1$ → ray bends **toward** normal (into slower medium)
- Vertical incidence ($\theta_1 = 0$): no bending at all

Units: $[\sin\theta/V] = 1/(\mathrm{m/s}) = \mathrm{s/m}$ ✓

Identical to Snell's law in optics — with V replacing c/n.

The ray parameter p

$$p = \frac{\sin\theta}{V} = \text{constant along the entire ray}$$

Physical meaning: horizontal component of the slowness vector $\mathbf{s} = \hat{n}/V$

Why conserved? Horizontal translational symmetry — properties vary only with depth. Same physics as conservation of horizontal momentum.

Through N layers: $p = \sin\theta_1/V_1 = \sin\theta_2/V_2 = \cdots = \sin\theta_N/V_N$

Worked example: three-layer model

A ray with $p = 0.0002$ s/m passes through:

Layer	V (m/s)	$\sin\theta = pV$	θ
1 (sediment)	2000	0.400	23.6°
2 (limestone)	4500	0.900	64.2°
3 (basement)	6000	1.200	**No real angle!**

The ray **cannot enter layer 3** — it is totally reflected.

Critical angle at layer 2→3: $\theta_c = \arcsin(4500/6000) = 48.6^\circ$

The ray arrives at 64.2° > 48.6° → post-critical.

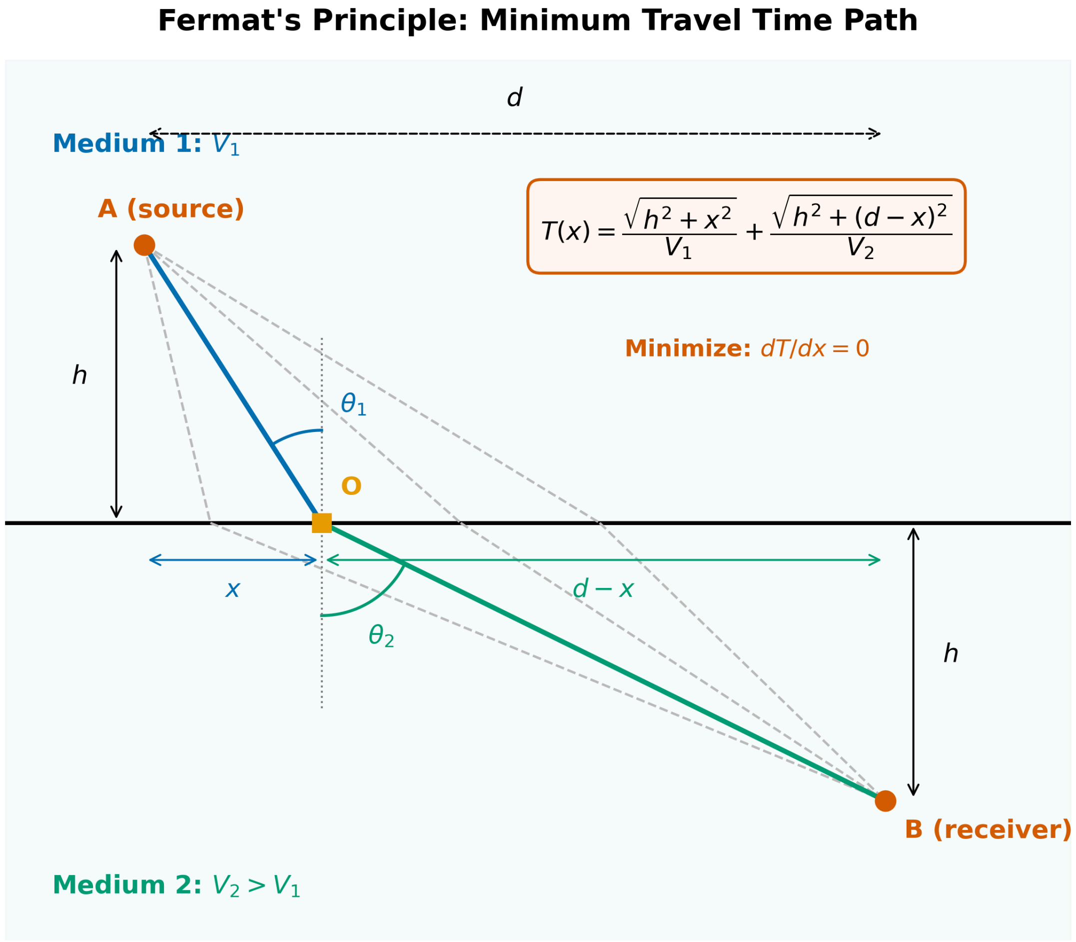
Fermat's principle of least time

The actual ray path between two points is the one with the shortest (stationary) travel time.

More general than Snell's law: works for curved interfaces, continuous gradients, and 3D media.

Snell's law is a *consequence* of Fermat's principle for flat interfaces.

Fermat's principle: the geometry



Fermat's principle: the calculus

Travel time from A to B via crossing point x:

$$T(x) = \frac{\sqrt{h^2 + x^2}}{V_1} + \frac{\sqrt{h^2 + (d - x)^2}}{V_2}$$

Minimize: set $dT/dx = 0$:

$$\frac{x}{V_1\sqrt{h^2 + x^2}} = \frac{d - x}{V_2\sqrt{h^2 + (d - x)^2}}$$

Recognize: $\sin\theta_1 = x/\sqrt{h^2 + x^2}$, $\sin\theta_2 = (d - x)/\sqrt{h^2 + (d - x)^2}$

$$\boxed{\frac{\sin\theta_1}{V_1} = \frac{\sin\theta_2}{V_2}}$$

Snell's law — from calculus, not geometry.

Two derivations, one law

Approach	Method	Strength
Huygens	Wavefront geometry at interface	Physical intuition — "see" the bending
Fermat	Minimize travel time via $dT/dx = 0$	Generalizes to curves, gradients, 3D

Both yield $p = \sin\theta/V = \text{constant.}$

The ray parameter p is the **fundamental invariant** of ray theory.

Reflection: The Other Half of Snell's Law

At every velocity contrast, energy **both refracts and reflects**.

The reflected ray stays in medium 1. Snell's law with $V_1 = V_1$:

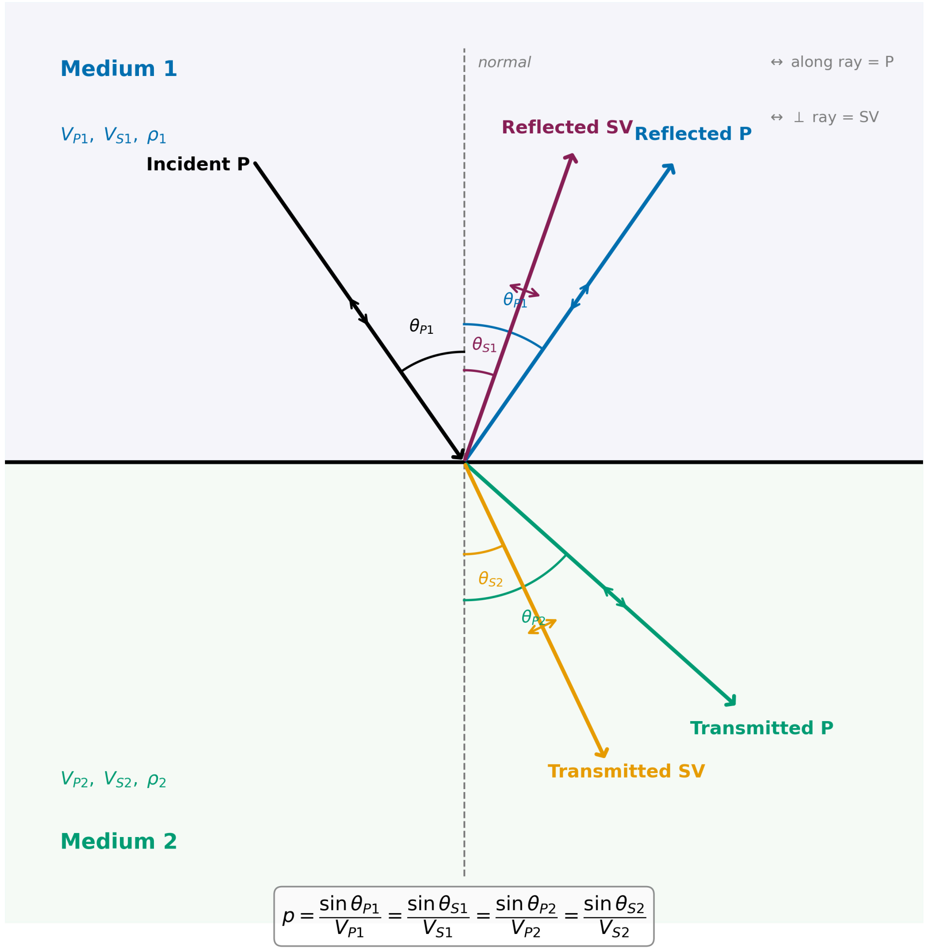
$$\frac{\sin\theta_i}{V_1} = \frac{\sin\theta_r}{V_1} \quad \Longrightarrow \quad \theta_r = \theta_i$$

The angle of reflection equals the angle of incidence.

This is symmetric about the normal — like a mirror.

Only when $Z_1 = Z_2$ does the reflected wave vanish entirely.

Mode Conversion: One P-Wave In, Four Waves Out



Since $V_S < V_P$: the converted S-wave is always **steeper** than the P-wave.

Note: SH waves do not convert — they are decoupled from P–SV.

How Much Reflects? Acoustic Impedance

Snell's law gives the **angles**. The **amplitudes** depend on **acoustic impedance**:

$$Z = \rho V \quad [\mathrm{kg/(m^2 \cdot s)}]$$

At **normal incidence** ($\theta = 0$), the reflection and transmission coefficients:

$$R = \frac{Z_2 - Z_1}{Z_2 + Z_1}, \qquad T = \frac{2\,Z_1}{Z_2 + Z_1}$$

Property	Meaning
$R > 0$	$Z_2 > Z_1$ — same polarity
$R < 0$	$Z_2 < Z_1$ — **polarity flip**
$R = 0$	$Z_1 = Z_2$ — no reflection
$R^2 + (Z_1/Z_2)\,T^2 = 1$	Energy conservation

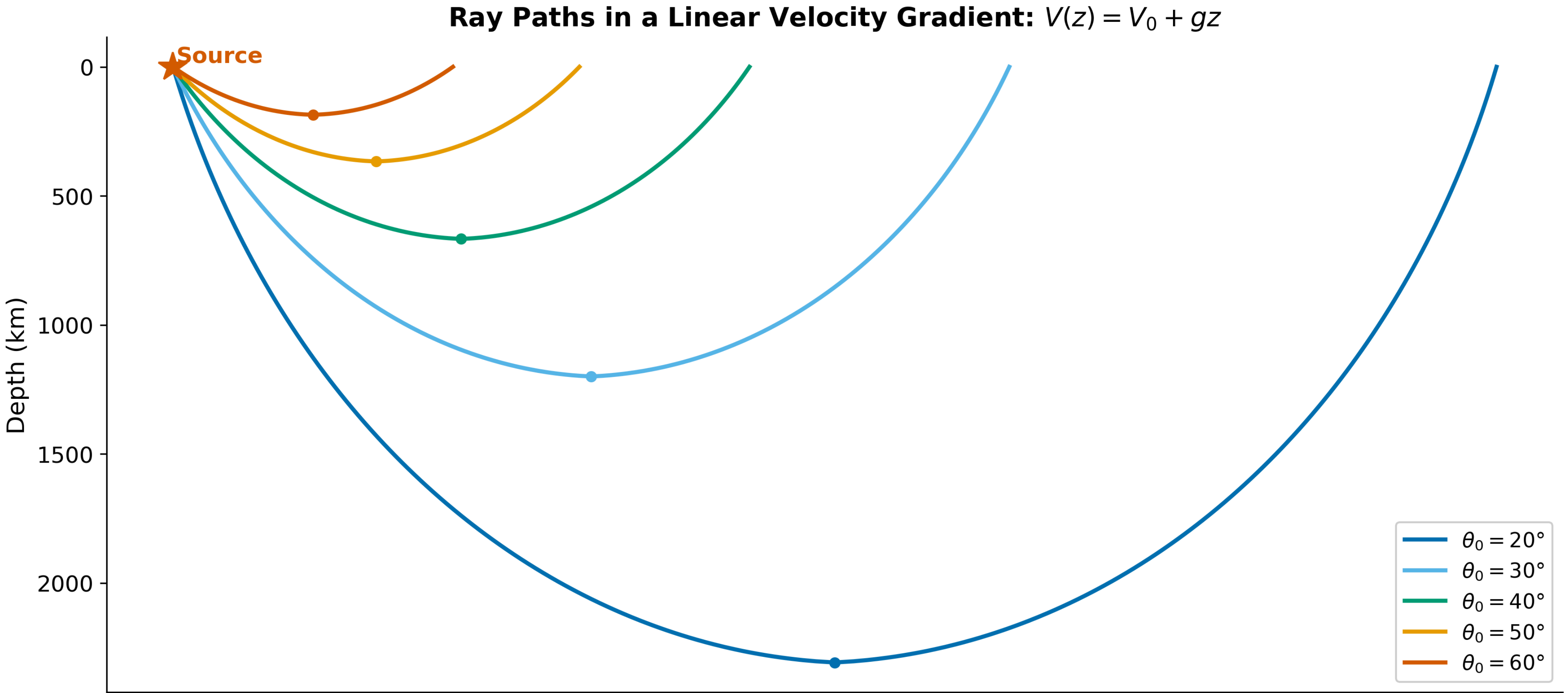
At oblique incidence → **Zoeppritz equations** (Lecture 8)

Rays in a velocity gradient

When V increases with depth ($V = V(z)$), rays **curve continuously**:

$p = \sin\theta(z)/V(z) =$ constant → as V increases, $\sin\theta$ increases → ray tilts toward horizontal

Turning depth: where $\sin\theta = 1$ (ray is horizontal), at $V(z_{\text{turn}}) = 1/p$



Rays in the real Earth

Steeper takeoff angle → smaller p → deeper penetration → greater epicentral distance

This is why:

- **Near stations** record shallow rays (refracted through the crust)
- **Distant stations** record deep rays (through the mantle)
- **Antipodal stations** record rays that traverse the core

The relationship $p(\Delta)$ — ray parameter vs. distance — is the **key to inverting Earth structure** from travel times.

The optical analogy

	Optics	Seismology
Law	$n_1 \sin\theta_1 = n_2 \sin\theta_2$	$\sin\theta_1 / V_1 = \sin\theta_2 / V_2$
"Slow" medium	Higher refractive index n	Lower velocity V
Bending	Toward normal in dense glass	Toward normal in slow rock

A **mirage** on a hot road = a seismic **turning ray** in a velocity gradient.

Same physics, different scale.

Critical angle and total reflection

When $V_2 > V_1$, increasing θ_i eventually makes $\sin\theta_2 = 1$ — the transmitted ray grazes the interface.

$$\theta_c = \arcsin\left(\frac{V_1}{V_2}\right)$$

For $\theta > \theta_c$: **total reflection** — no transmitted energy, $|R| = 1$

At exactly θ_c: the wave along the interface radiates a **head wave** back into medium 1 at angle θ_c.

Head waves travel at V_2 → the foundation of **seismic refraction** (Lectures 6–7).

Why it matters: earthquake location

Every PNSN earthquake location depends on:

1. A **velocity model** $V(z)$ for the PNW crust and mantle
2. **Snell's law** applied at every layer boundary to trace rays
3. Travel times computed from ray paths

Errors in the velocity model $\rightarrow$ errors in location (5–10 km in Cascadia with 1D models).

The **M9 Project** uses 3D ray tracing through the Community Velocity Model to simulate ground motion for a future Cascadia M9.

AI as a reasoning partner

Try this prompt:

> *"Derive Snell's law from Fermat's principle. Start from* $T(x) = \sqrt{h^2 + x^2}/V_1 + \sqrt{h^2 + (d - x)^2}/V_2$*, take* dT/dx*, set to zero, and show every step."*

Evaluate the AI's response against today's derivation:

- Does it correctly differentiate both terms?
- Does it recognize $\sin\theta = x/\sqrt{h^2 + x^2}$ from the geometry?
- Common error: dropping the negative sign in $(d - x)$

Concept Check

1. A ray enters sandstone ($V = 3000$ m/s) from sediment ($V = 1500$ m/s) at $\theta_1 = 20^\circ$. Find θ_2. Is the ray bending toward or away from the normal?
2. Sketch wavefronts for a wave propagating downward through a medium where V increases linearly with depth. Are they flat, curved up, or curved down?
3. A ray with $p = 1.5 \times 10^{-4}$ s/m enters a medium where V increases from 4000 to 8000 m/s. At what velocity does the ray turn? What is the takeoff angle at the surface?

Next time

Lecture 6: Seismic Refraction I

The head wave we just introduced is the key observable. Lecture 6 derives the travel-time equations for head waves and shows how to invert slopes and intercepts for layer velocity and thickness — the method Mohorovičić used to discover the crust–mantle boundary.

Direct waves · head waves · travel-time curves · crossover distance · the Moho

Discussion (Wed): Radar eyes on ice — applying today's physics to GPR